

Differential Complexes and Stratified Pro-Modules

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ABSTRACT. In this paper we introduce the category of stratified Pro-modules and the notion of induced object in this category. We propose a translation of Morihiko Saito equivalence results ([S.2]) using the dual language of Pro-objects. So we prove an equivalence between the derived category of stratified Pro-modules and the category of Pro-differential complexes. We also supply a comparison with the notion of Crystal in Pro-module (introduced by P. Deligne in 1960).

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Introduction.

Let X be a smooth separated noetherian scheme of finite type over \mathbb{C} . We first note that a differential operator of finite order $m \in \mathbb{N}$ (between \mathcal{O}_X -modules) can be defined in two different ways: the first using induced right $\mathcal{D}_{X,m}$ -modules, as done by Morihiko Saito in [S.2], the second using the sheaf of principal parts \mathcal{P}_X^m . Thus, any differential complex \mathcal{L}^\bullet admits two “linearized” versions. The first is given by M. Saito’s functor $\mathrm{DR}_X^{-1}(\mathcal{L}^\bullet) := \mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X$ in the category of right \mathcal{D}_X -modules while the other is given by Grothendieck’s formalization functor $\mathrm{Q}_X^0 := \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet$ in the category of stratified Pro-modules.

In [S.2] M. Saito proved the equivalence between the derived category of right \mathcal{D}_X -modules (quasi-coherent as \mathcal{O}_X -ones) and a suitable localized category of differential complexes. Our aim is to prove a “dual” version of this equivalence replacing quasi-coherent right \mathcal{D}_X -modules by Pro-coherent stratified ones. The main idea is that of using the Grothendieck formalization functor Q_X^0 instead of Saito’s DR_X^{-1} . On the other hand, a functor DR_X is always defined on stratified objects simply by taking horizontal sections. Suitably localizing these functors gives an equivalence between the derived category of Stratified Pro-coherent modules and that of Pro-differential complexes (suitably localized). We also define a category of Ω_X^\bullet -modules in Pro-object (suitable localized) and we prove an equivalence with that of stratified Pro-modules (as done in the dual case in [F1]).

In the last section we interpret stratified Pro-coherent modules as objects in the Crystalline site. In particular we prove that the category of stratified Pro-coherent modules is equivalent to the category of “crystals in Pro-modules”. This last notion was first introduced by P. Deligne in a cycle of lectures he gave at IHES. There Deligne proposed the notion of “Crystals in Pro-modules” attached to algebraic constructible sheaves on an analytic space X^{an} . Moreover he proved that the category of “regular crystals in Pro-modules” is equivalent to that of “algebraic” constructible sheaves on X^{an} (unfortunately this work was not published). By Deligne’s equivalence theorem we obtain the equivalence between the derived category of regular stratified Pro-coherent modules and that of “algebraic” constructible sheaves, and thus a sort of Riemann-Hilbert correspondence.

As noted above the notion of stratified Pro-coherent module is dual to that of quasi-coherent right \mathcal{D} -module. In a work in progress we expect to prove an anti-equivalence of categories between the category of perfect \mathcal{D} -complexes and that of perfect complexes of stratified Pro-modules. This anti-equivalence is compatible with the duality in the category of differential complexes. In particular when any object of a differential complex is coherent on \mathcal{O}_X , the notion of \mathcal{D}_X -qis (see [S.2]) is equivalent to that of Q_X^0 -qis.

I would like to thank Pierre Berthelot for his suggestions and for his notes on Deligne lectures on “Cristaux discontinues”. I would also like to thank Maurizio Cailotto and Anne Virrion for the improvements they suggested to me during the preparation of this work.

1. Pro-Coherent \mathcal{O}_X -Modules

We briefly recall some results on the category of Pro-coherent \mathcal{O}_X -modules.

1.1. Definition. ([SGAIII, 1]). By definition the category $\text{Pro}(\text{Coh}(\mathcal{O}_X)) := \text{Ind}(\text{Coh}(\mathcal{O}_X)^\circ)^\circ$. Objects are filtering projective systems of coherent \mathcal{O}_X -modules, while morphisms between two such objects $\mathcal{F}_I, \mathcal{G}_J$ are $\text{Hom}_{\text{Pro}}(\mathcal{F}_I, \mathcal{G}_J) = \varprojlim_I \varinjlim_J \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{G}_j)$. For brevity we will use the notation $\nu(\mathcal{O}_X)$ for $\text{Pro}(\text{Coh}(\mathcal{O}_X))$ and $\mu(\mathcal{O}_X)$ for $\text{Ind}(\text{Coh}(\mathcal{O}_X))$.

1.2. Remark. Given a noetherian scheme X over \mathbb{C} , the category $\mu(\mathcal{O}_X)$ is equivalent to the category of quasi-coherent \mathcal{O}_X -modules (denoted by $\text{QCoh}(\mathcal{O}_X)$) see [RD, appendix]. Moreover any object in $\mu(\mathcal{O}_X)$ may be represented as an inductive system whose transition morphisms are injective maps.

1.3. Remark. The functor tensor product:

$$\begin{aligned} - \otimes_{\mathcal{O}_X} - : \quad \text{Coh}(\mathcal{O}_X) \times \text{Coh}(\mathcal{O}_X) &\longrightarrow \text{Coh}(\mathcal{O}_X) \\ (\mathcal{F}, \mathcal{G}) &\longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \end{aligned}$$

extends to the procategory:

$$\begin{aligned} - \otimes_{\mathcal{O}_X} - : \quad \text{Pro}(\text{Coh}(\mathcal{O}_X)) \times \text{Pro}(\text{Coh}(\mathcal{O}_X)) &\longrightarrow \text{Pro}(\text{Coh}(\mathcal{O}_X)) \\ (\{\mathcal{F}_i\}_I, \{\mathcal{G}_j\}_J) &\longmapsto \{\mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{G}_j\}_{I \times J}. \end{aligned}$$

1.4. Remark. Let \mathcal{C} be an abelian category, then $\text{Pro}(\mathcal{C})$ is abelian (see [AM, appendix]). Moreover if \mathcal{C} has enough injectives the same is true for $\text{Pro}(\mathcal{C})$ (see [J], or [AM]). This result mainly concerns the description of $\text{Ker}(f)$ and $\text{Coker}(f)$, where f is a morphism in $\text{Pro}(\mathcal{C})$, done in [AM]. So $\nu(\mathcal{O}_X)$ is an abelian category. Let $\text{Pro}(\text{QCoh}(\mathcal{O}_X))$ be the Pro-category of quasi-coherent \mathcal{O}_X -modules. It is an abelian category and it has enough injectives (since $\text{QCoh}(\mathcal{O}_X)$ has enough injectives), moreover $\nu(\mathcal{O}_X)$ is a full thick subcategory of $\text{Pro}(\text{QCoh}(\mathcal{O}_X))$. In fact $\text{Coh}(\mathcal{O}_X)$ is a full thick subcategory of $\text{QCoh}(\mathcal{O}_X)$, and it is easy to prove that the same is true for their Pro-categories ([F2]).

1.5. Remark. Let denote by $D_{\nu(\mathcal{O}_X)}^+(\text{Pro}(\text{QCoh}(\mathcal{O}_X)))$ the derived category of $\text{Pro}(\text{QCoh}(\mathcal{O}_X))$ with cohomology bounded below and in $\nu(\mathcal{O}_X)$. Then $D_{\nu(\mathcal{O}_X)}^+(\text{Pro}(\text{QCoh}(\mathcal{O}_X)))$ is equivalent to the derived category $D^+(\nu(\mathcal{O}_X))$.

2. Stratified Pro-Modules

In this paper we consider X a smooth algebraic variety over \mathbb{C} . We will denote by $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ the projective system of sheaves of principal parts ([EGA IV]), by $q_m : \mathcal{P}_X^m \longrightarrow \mathcal{O}_X$ the map induced by the diagonal embedding $X \longrightarrow X \times X$ and by $q_{m,n} : \mathcal{P}_X^m \longrightarrow \mathcal{P}_X^n$ ($m \geq n$) the maps of the projective system $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$. By definition $\mathcal{D}_{X,m} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_X^m, \mathcal{O}_X)$ and $\mathcal{D}_X = \varinjlim_{m \in \mathbb{N}} \mathcal{D}_{X,m}$ is the sheaf of differential operators. We denote by Ω_X^\bullet the De Rham complex of algebraic differential forms and by $\Theta_X^{-i} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^i, \mathcal{O}_X)$ its dual. Moreover let $d := d_X$ be the dimension of X ; we denote by $\omega_X := \Omega_X^d$ the sheaf of differential forms of maximum degree.

2.1. Definition. ([BeO; 2.10]). Let X be a smooth separated noetherian scheme of finite type over \mathbb{C} and let \mathcal{F} be an \mathcal{O}_X -module. A stratification on \mathcal{F} is a collection (one for any $n \in \mathbb{N}$) of \mathcal{P}_X^n -linear isomorphisms

$$\varepsilon_{\mathcal{F},n} : \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n$$

such that $\varepsilon_{\mathcal{F},n}$ and $\varepsilon_{\mathcal{F},m}$ are compatible via $q_{n,m}$ for each $m \leq n$, the map $\varepsilon_{\mathcal{F},0}$ is the identity, and the cocycle condition holds.

2.2. Proposition. ([BeO; 2.11]). Let \mathcal{F} be an \mathcal{O}_X -module, the following are equivalent:

i) there is a collection of maps

$$s_{\mathcal{F},n} : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n$$

“right” \mathcal{O}_X -linear such that $s_{\mathcal{F},0} = id_{\mathcal{F}}$, $(id_{\mathcal{F}} \otimes_{\mathcal{O}_X} q_{m,n}) \circ s_{\mathcal{F},m} = s_{\mathcal{F},n}$ and $(s_{\mathcal{F},n} \otimes_{\mathcal{O}_X} id_{\mathcal{P}_X^m}) \circ s_{\mathcal{F},m} = (id_{\mathcal{F}} \otimes_{\mathcal{O}_X} \delta^{n,m}) \circ s_{\mathcal{F},m+n}$ (see [EGA IV,16.8.9.1] for the definition of $\delta^{m,n}$);

i') there is a collection of maps

$$s'_{\mathcal{F},n} : \mathcal{F} \longrightarrow \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F}$$

“left” \mathcal{O}_X -linear such that $s'_{\mathcal{F},0} = id_{\mathcal{F}}$, $(q_{m,n} \otimes_{\mathcal{O}_X} id_{\mathcal{F}}) \circ s'_{\mathcal{F},m} = s'_{\mathcal{F},n}$ and $(id_{\mathcal{P}_X^m} \otimes_{\mathcal{O}_X} s'_{\mathcal{F},n}) \circ s'_{\mathcal{F},m} = (\delta^{n,m} \otimes_{\mathcal{O}_X} id_{\mathcal{F}}) \circ s'_{\mathcal{F},m+n}$;

ii) \mathcal{F} is a stratified module;

iii) \mathcal{F} is a left \mathcal{D}_X -module, where \mathcal{D}_X is the sheaf of rings of differential operators.

Proof. $i) \Leftrightarrow ii)$ [BeO, 2.11].

$i) + ii) \Rightarrow i')$ and $i') + ii) \Rightarrow i)$.

iii) $\Leftrightarrow i)$ Let $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{m_{\mathcal{F}}} \mathcal{F}$ be the multiplication on the \mathcal{D}_X -module \mathcal{F} then

$$\begin{aligned} m_{\mathcal{F}} \in \text{Hom}_{\mathcal{O}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{F}) &= \varprojlim_{m \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\mathcal{D}_{X,m} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{F}) \\ &= \varprojlim_{m \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^m). \end{aligned}$$

The associative diagram induces the diagram for co-associativity, and the identity diagram induces that of the co-identity. \square

Stratified \mathcal{O}_X -modules form a category which we denote by $\mathcal{O}_X\text{-Strat}$. Morphisms are \mathcal{O}_X -linear maps which respect the stratifications. We are now interested only in coherent objects so $\text{Coh}(\mathcal{O}_X)\text{-Strat}$ will denote the full subcategory of $\mathcal{O}_X\text{-Strat}$ whose objects are coherent.

We want to extend this category to Pro-objects, in order to obtain a category dual to that of quasi-coherent (so $\text{Ind}(\text{Coh}(\mathcal{O}_X))$) right \mathcal{D}_X -modules.

The naive way would be that of taking simply the Pro-category $\text{Pro}(\mathcal{O}_X\text{-Strat})$, but in this way we obtain Pro-objects which have a stratification at any “level” while we need a larger category, that of stratified Pro-objects defined as follow.

2.3. Definition. Let $\nu(\mathcal{P}_X)$ be the category whose objects are Pro-coherent \mathcal{O}_X -modules $\{\mathcal{F}_h\}_H$ endowed with a stratification that is a morphism of Pro-objects

$$\{\mathcal{F}_h\}_H \xrightarrow{s_{\{\mathcal{F}_h\}_H}} \{\mathcal{F}_h\}_H \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}}$$

which make the co-identity diagram

$$(2.3.1) \quad \begin{array}{ccc} \{\mathcal{F}_h\}_H & \xrightarrow{s_{\{\mathcal{F}_h\}_H}} & \{\mathcal{F}_h\}_H \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \\ & \searrow id_{\{\mathcal{F}_h\}_H} & \downarrow id_{\{\mathcal{F}_h\}_H} \otimes \{q_m\}_{\mathbb{Z}} \\ & & \{\mathcal{F}_h\}_H \end{array}$$

and the co-associative one

$$(2.3.2) \quad \begin{array}{ccc} \{\mathcal{F}_h\}_H & \xrightarrow{s_{\{\mathcal{F}_h\}_H}} & \{\mathcal{F}_h\}_H \otimes \{\mathcal{P}_X^m\}_{\mathbb{Z}} \\ \downarrow s_{\{\mathcal{F}_h\}_H} & & \downarrow s_{\{\mathcal{F}_h\}_H} \otimes id_{\{\mathcal{P}_X^m\}_{\mathbb{Z}}} \\ \{\mathcal{F}_h\}_H \otimes \{\mathcal{P}_X^m\}_{\mathbb{Z}} & \xrightarrow{id_{\{\mathcal{F}_h\}_H} \otimes s_{\{\mathcal{P}_X^m\}_{\mathbb{Z}}}} & \{\mathcal{F}_h\}_H \otimes \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes \{\mathcal{P}_X^m\}_{\mathbb{Z}} \end{array}$$

commutative. (This is simply the category of $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ -co-modules in the category $\nu(\mathcal{O}_X)$).

By definition $s_{\{\mathcal{P}_X^m\}_{\mathbb{Z}}} := \{\delta^{m,n}\}_{\mathbb{Z} \times \mathbb{Z}}$ is the map inducing the stratification on the “right” $\{\mathcal{P}_X^m\}_{\mathbb{Z}} = p_1^*(\mathcal{O}_X)$ (see [BeO, Remark 2.13], [G]). A morphism of Pro-objects $f : \{\mathcal{F}_h\}_H \longrightarrow \{\mathcal{G}_k\}_K$ is a morphism

in $\nu(\mathcal{P}_X)$ if and only if the diagram

$$(2.3.3) \quad \begin{array}{ccc} \{\mathcal{F}_h\}_H & \xrightarrow{f} & \{\mathcal{G}_k\}_K \\ \downarrow s_{\{\mathcal{F}_h\}_H} & & \downarrow s_{\{\mathcal{G}_k\}_K} \\ \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes \{\mathcal{F}_h\}_H & \xrightarrow{id \otimes f} & \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes \{\mathcal{G}_k\}_K \end{array}$$

commutes. We denote by $\mathcal{H}om_{\mathcal{P}_X}(\{\mathcal{F}_h\}_H, \{\mathcal{G}_k\}_K)$ (or $\mathcal{H}om_{\text{Strat}}(\{\mathcal{F}_h\}_H, \{\mathcal{G}_k\}_K)$) the sheafified version of the set of morphisms in $\nu(\mathcal{P}_X)$.

We denote by $C^*(\mathcal{P}_X)$ (resp. $K^*(\mathcal{P}_X)$, resp. $D^*(\mathcal{P}_X)$) with $*$ $\in \{+, -, b\}$ the category of complexes (bounded below, bounded above, bounded) (resp. up to homotopy, resp. up to quasi-isomorphisms) in $\nu(\mathcal{O}_X)$.

2.4. Remark. [BeO; 2.2, 2.3]. In our setting X is smooth, hence the sheaves $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ are locally free of finite type. A base of \mathcal{P}_X^m (for both left and right \mathcal{O}_X -module structures) is $\{\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} | \alpha_1 + \cdots + \alpha_d \leq m; \alpha_i \in \mathbb{N}\}$ where $\xi_i := 1 \otimes x_i - x_i \otimes 1$ and $\mathbb{I} = 1 \otimes 1$ in local coordinates. By this description the “right” stratification on the Pro-system $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ is given by the morphisms

$$\begin{array}{lll} \delta^{m,p} : \mathcal{P}_X^m & \longrightarrow & \mathcal{P}_X^{m-p} \otimes_{\mathcal{O}_X} \mathcal{P}_X^p \\ \mathbb{I} & \longmapsto & \mathbb{I} \otimes \mathbb{I} \\ \xi_i & \longmapsto & \mathbb{I} \otimes \xi_i + \xi_i \otimes \mathbb{I} \\ \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} & \longmapsto & \delta^{m,p}(\xi_1)^{\alpha_1} \cdots \delta^{m,p}(\xi_d)^{\alpha_d} \quad \forall \alpha_1, \dots, \alpha_d \in \mathbb{N}, \alpha_1 + \cdots + \alpha_d \leq m \end{array}$$

Moreover the sheaf Ω_X^1 is simply the sub-sheaf of \mathcal{P}_X^1 generated by ξ_i for $i = 1, \dots, d$.

2.5. Definition. A stratified Pro-module is induced if it is isomorphic to $\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}$, for some $\mathcal{L} \in \nu(\mathcal{O}_X)$, endowed with the stratification induced by the canonical one on $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ (see [G, 6.3]). We denote by $\nu_i(\mathcal{P}_X)$ the full subcategory of $\nu(\mathcal{P}_X)$ whose objects are induced. We denote by $C_i^b(\mathcal{P}_X)$ (resp. $K_i^b(\mathcal{P}_X)$, resp. $D_i^b(\mathcal{P}_X)$) the category of bounded complexes (resp. up to homotopy, resp. up to quasi-isomorphisms) in $\nu_i(\mathcal{O}_X)$.

2.6. Proposition. The category $\nu(\mathcal{P}_X)$ is an abelian category, small filtering projective limits are representable and exact. The forgetful functor

$$for : \nu(\mathcal{P}_X) \longrightarrow \nu(\mathcal{O}_X)$$

has a right adjoint

$$Q_X^0 := \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} - : \begin{array}{ccc} \nu(\mathcal{O}_X) & \longrightarrow & \nu(\mathcal{P}_X) \\ \{\mathcal{F}_h\}_H & \longmapsto & \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \{\mathcal{F}_h\}_H \end{array}$$

which takes image into $\nu_i(\mathcal{P}_X)$.

Proof. Kernels and cokernels in $\nu(\mathcal{P}_X)$ are those of $\nu(\mathcal{O}_X)$ endowed with the induced stratification and for any morphism f in $\nu(\mathcal{P}_X)$, the image of f is isomorphic to its co-image. So $\nu(\mathcal{P}_X)$ is an abelian category and the forgetful functor is exact. Small filtering limits are representable and exact because they are representable in $\nu(\mathcal{O}_X)$ and they have canonical stratifications.

The map

$$\begin{array}{ccc} \mathcal{H}om_{\text{Strat}}(\{\mathcal{F}_h\}_H, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \{\mathcal{G}_k\}_K) & \xrightarrow{\alpha} & \mathcal{H}om_{\nu(\mathcal{O}_X)}(for(\{\mathcal{F}_h\}_H), \{\mathcal{G}_k\}_K) \\ f & \longmapsto & (\{q_m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} id_{\{\mathcal{G}_k\}_K}) \circ f \end{array}$$

(co-extension of scalars) is a bijection whose inverse is the map

$$\begin{array}{ccc} \mathcal{H}om_{\nu(\mathcal{O}_X)}(for(\{\mathcal{F}_h\}_H), \{\mathcal{G}_k\}_K) & \xrightarrow{\beta} & \mathcal{H}om_{\{\mathcal{P}_X^m\}_{\mathbb{Z}}}(\{\mathcal{F}_h\}_H, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \{\mathcal{G}_k\}_K) \\ g & \longmapsto & (id_{\{\mathcal{P}_X^m\}_{\mathbb{Z}}} \otimes g) \circ \{s'_{\mathcal{F},m}\}_{\mathbb{Z}}. \end{array}$$

Clearly $\alpha(f)$ is a morphism in $\nu(\mathcal{O}_X)$; on the other hand in order to prove that $\beta(g)$ respects the stratifications it is sufficient to remark that Q_X^0 is a functor so the map $(id_{\{\mathcal{P}_X^m\}_{\mathbb{Z}}} \otimes g)$ respects the canonical stratifications on $\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} -$. \square

2.7. Remark. Proposition 2.6 also holds true on replacing $\nu(\mathcal{O}_X)$ by the category of Pro-quasi-coherent \mathcal{O}_X -modules $\text{Pro}(\mu(\mathcal{O}_X))$, and $\nu(\mathcal{P}_X)$ by the category of stratified Pro-quasi-coherent \mathcal{O}_X -modules denoted $\text{Pro}(\mu(\mathcal{P}_X))$.

2.8. Corollary. Any object in $\text{Pro}(\mu(\mathcal{P}_X))$ induced by an injective object in $\text{Pro}(\mu(\mathcal{O}_X))$ is injective. Moreover $\text{Pro}(\mu(\mathcal{P}_X))$ has enough injectives.

Proof. Let \mathcal{E} be an injective Pro-quasi-coherent \mathcal{O}_X -module, then the functor

$$\mathcal{H}om_{\text{Strat}}(-, \{\mathcal{P}_X^m\} \otimes_{\mathcal{O}_X} \mathcal{E}) \cong \mathcal{H}om_{\text{Pro}(\mu(\mathcal{O}_X))}(for(-), \mathcal{E})$$

is exact because $for(-) : \text{Pro}(\mu(\mathcal{P}_X)) \rightarrow \text{Pro}(\mu(\mathcal{O}_X))$ is exact and \mathcal{E} is injective.

For each $\mathcal{N} \in \text{Pro}(\mu(\mathcal{P}_X))$, there exists $\mathcal{I} \in \text{Pro}(\mu(\mathcal{O}_X))$ and an injective map $i : for(\mathcal{N}) \hookrightarrow \mathcal{I}$ in $\text{Pro}(\mu(\mathcal{O}_X))$. Then the map $\beta(i) : \mathcal{N} \rightarrow \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective map in $\text{Pro}(\mu(\mathcal{P}_X))$ (and $\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{I}$ is injective in $\text{Pro}(\mu(\mathcal{P}_X))$). \square

2.9. Corollary. Derived co-extension of scalars.

Let $\mathcal{F} \in \nu(\mathcal{P}_X)$ and $\mathcal{G} \in \nu(\mathcal{O}_X)$:

$$\mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{F}, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{G}) \xrightarrow{\cong} \mathbf{R}\mathcal{H}om_{\nu(\mathcal{O}_X)}(for(\mathcal{F}), \mathcal{G});$$

is a quasi-isomorphism.

Proof. Let denote by $E^\bullet(\mathcal{G})$ an injective resolution of \mathcal{G} in $\text{Pro}(\mu(\mathcal{O}_X))$. Then

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{F}, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{G}) &= \mathcal{H}om_{\text{Strat}}(\mathcal{F}, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} E^\bullet(\mathcal{G})) \cong \\ &\cong \mathcal{H}om_{\text{Pro}(\mu(\mathcal{O}_X))}(for(\mathcal{F}), E^\bullet(\mathcal{G})) = \\ &= \mathbf{R}\mathcal{H}om_{\nu(\mathcal{O}_X)}(for(\mathcal{F}), \mathcal{G}). \end{aligned}$$

\square

Let consider the De Rham functor

$$\mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, -) : D^b(\nu(\mathcal{P}_X)) \rightarrow D^b(\mathbb{C}_X)$$

where \mathbb{C}_X denotes the category of sheaves in \mathbb{C} -vector spaces. Then for any $\mathcal{M} \in D^b(\nu(\mathcal{P}_X))$ deriving it in $\text{Pro}(\mu(\mathcal{P}_X))$ we have:

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, \mathcal{M}) &\cong \mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{D}_X \otimes \Theta_X, \mathcal{M}) \cong \\ &\cong \mathcal{H}om_{\text{Strat}}(\mathcal{D}_X \otimes \Theta_X, \mathcal{M}) \cong \\ &\cong \Omega_X \otimes \mathcal{M} \end{aligned}$$

The complex $\Omega_X \otimes \mathcal{M}$ is a complex of Pro-coherent- \mathcal{O}_X -modules but its differentials are not \mathcal{O}_X -linear.

In the following we will define the category $\nu(\mathcal{O}_X)\text{-Diff}_X$ wherein the functor $\mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, -)$ has its image in a fully faithful way. So the De Rham functor will have its image in a suitable localization of $\nu(\mathcal{O}_X)\text{-Diff}_X$.

2.10. Theorem. Induced stratified Pro-modules are acyclic for the functor $\mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, -)$. For \mathcal{M}, \mathcal{N} such modules

$$\mathcal{M}^\nabla := \mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, \mathcal{M}) = \mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, \mathcal{M})$$

and the morphism

$$(2.10.1) \quad \mathcal{H}om_{\text{Strat}}(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{H}om_{\mathbb{C}_X}(\mathcal{M}^\nabla, \mathcal{N}^\nabla)$$

is injective.

Proof. By hypothesis there exist $\mathcal{L}, \mathcal{L}'$ in $\nu(\mathcal{O}_X)$ such that $\mathcal{M} \cong \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}$ and $\mathcal{N} \cong \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}'$. Then

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, \mathcal{M}) &= \mathbf{R}\mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}) \cong \\ &\cong \mathbf{R}\mathcal{H}om_{\nu(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{L}) \cong \\ &\cong \mathcal{H}om_{\nu(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{L}) \cong \\ &\cong \mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}) = \\ &= \mathcal{H}om_{\text{Strat}}(\mathcal{O}_X, \mathcal{M}) \end{aligned}$$

which proves the first assertion.

For the second statement let consider the map

$$(2.10.2) \quad \begin{aligned} \mathcal{H}om_{\text{Strat}}(\mathcal{M}, \mathcal{N}) &= \mathcal{H}om_{\text{Strat}}(\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}') \cong \\ &\cong \mathcal{H}om_{\nu(\mathcal{O}_X)}(\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}, \mathcal{L}') \longrightarrow \mathcal{H}om_{\mathbb{C}_X}(\mathcal{L}, \mathcal{L}') \end{aligned}$$

obtained by composition with the stratification morphism $s'_{\mathcal{L}} : \mathcal{L} \rightarrow \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}$. It is injective because the image of $s'_{\mathcal{L}}$ generates $\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}$ as Pro-coherent- \mathcal{O}_X -module.

We note that this theorem is the analogue (for Pro-objects) of Saito's Lemma [S.2; 1.2]. \square

3. Differential Complexes of Pro-modules

3.1. Definition. Let $\{\mathcal{L}_i\}_I$ and $\{\mathcal{L}'_j\}_J$ be two Pro-coherent \mathcal{O}_X -modules. The sheaf of differential operators, which we denoted by $\mathcal{H}om_{\text{Diff}}(\{\mathcal{L}_i\}_I, \{\mathcal{L}'_j\}_J)$, is the image of the injective map (2.10.2). So

$$\begin{aligned} \mathcal{H}om_{\text{Diff}_X}(\{\mathcal{L}_i\}_I, \{\mathcal{L}'_j\}_J) &:= \mathcal{H}om_{\text{Strat}}(\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}, \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}') \cong \\ &\cong \mathcal{H}om_{\nu(\mathcal{O}_X)}(\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \{\mathcal{L}_i\}_I, \{\mathcal{L}'_j\}_J) := \\ &:= \lim_{\leftarrow J} \lim_{\rightarrow I} \lim_{\rightarrow \mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_X^m \otimes_{\mathcal{O}_X} \mathcal{L}_i, \mathcal{L}'_j) \cong \\ &\cong \lim_{\leftarrow J} \lim_{\rightarrow I} \mathcal{H}om_{\text{Diff}_X}(\mathcal{L}_i, \mathcal{L}'_j). \end{aligned}$$

We recall that for $\mathcal{F} \in \text{Coh}(\mathcal{O}_X)$ and \mathcal{G} an \mathcal{O}_X -module, the sheaf $\mathcal{H}om_{\text{Diff}_X}(\mathcal{F}, \mathcal{G})$ is isomorphic to $\varinjlim_{\mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_X^m \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \cong \varinjlim_{\mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_{X,m})$.

We denote by $\nu(\mathcal{O}_X)\text{-Diff}_X$ the additive category whose objects are Pro-coherent- \mathcal{O}_X -modules and whose morphisms are differential operators (sometimes called differential complexes). We have a functor

$$\begin{aligned} Q_X^0 : \nu(\mathcal{O}_X)\text{-Diff}_X &\longrightarrow \nu_i(\mathcal{P}_X^\bullet) \\ \mathcal{L} &\longmapsto \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L} \end{aligned}$$

which extends that of Proposition 2.6. This functor was firstly introduced by A. Grothendieck in [G; 6.2] and it is called the formalization functor or linearization. By (2.10.1) it is an equivalence of categories.

3.2. Remark. If we restrict the formalization functor to differential complexes \mathcal{L} whose objects are coherent \mathcal{O}_X -modules, then the Pro-objects $Q_X^0(\mathcal{L})$ are always of Artin-Rees type. Moreover any morphism of Pro-objects between two such objects is necessarily of Artin-Rees type (see [G; 6.2]).

3.3. Definition. Let $C^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ be the category of bounded complexes in $\nu(\mathcal{O}_X)\text{-Diff}_X$. Let $D^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ be the category obtained from $C^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ by inverting Q_X^0 -quasi-isomorphisms. This is a triangulated category with the usual shift functor and distinguished triangles those induced by the usual mapping cones. We remark that this localizing procedure was first introduced in [AB, Appendix C] following an idea of P. Berthelot.

We obtain a localized equivalence of categories

$$Q_X^0 : D^b(\nu(\mathcal{O}_X), \text{Diff}_X) \longrightarrow D_i^b(\mathcal{P}_X^\bullet).$$

This functor would be the “dual” of Saito $\widetilde{\text{DR}}_X^{-1}$ functor.

3.4. Remark. The morphism (2.10.1) is induced by the following commutative diagrams which are adjoint to those of \mathcal{D}_X -modules in the smooth case ([S.2; (1.4.1)]):

$$(3.4.1) \quad \begin{array}{ccc} \mathcal{L} & \xrightarrow{P} & \mathcal{L}' \\ d^1 \otimes \text{id}_{\mathcal{L}} \downarrow & & \downarrow d^1 \otimes \text{id}_{\mathcal{L}'} \\ \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L} & \xrightarrow{Q_X^0(P)} & \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}' \end{array} \quad \begin{array}{ccc} \mathcal{L} & \xrightarrow{P} & \mathcal{L}' \\ d^1 \otimes \text{id}_{\mathcal{L}} \downarrow & \nearrow Q_X^0(P) & \downarrow d^1 \otimes \text{id}_{\mathcal{L}'} \\ \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L} & & \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{L}' \end{array}$$

for any $Q_X^0(P) \in \text{Hom}_{\text{Strat}}(Q_X^0(\mathcal{L}), Q_X^0(\mathcal{L}')) \cong \text{Hom}_{\nu(\mathcal{O}_X)}(Q_X^0(\mathcal{L}), \mathcal{L}')$. The map $d_1 : \mathcal{O}_X \longrightarrow \mathcal{P}_X^m$ is that induced by the second projection of $X \times X$ into X .

3.5. Definition. An object in $D^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ is said to be perfect if it is locally isomorphic to a bounded complex whose elements are locally free \mathcal{O}_X -modules of finite rank. We denote by $D_p^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ the category of bounded perfect complexes in $D^b(\nu(\mathcal{O}_X), \text{Diff}_X)$. Then any object in $D_p^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ may be represented as an object in $C^b(\text{Coh}(\mathcal{O}_X), \text{Diff}_X)$ (see [L] for definition of perfect objects).

3.6. Definition. Herrera-Lieberman differential complexes.

([HL, §2] or [Be, II.5]). Let $C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ denote the category of bounded complexes of differential operators of order at most one, that is:

- i) the objects of $C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ are complexes whose terms are Pro-coherent- \mathcal{O}_X -modules and whose differentials are differential operators of order at most one;
- ii) morphisms between such complexes are morphisms of complexes which are \mathcal{O}_X -linear maps.

The category $C_1^b(\text{Coh}(\mathcal{O}_X), \text{Diff}_X)$ is the full subcategory of $C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ whose objects are coherent modules.

We denote by $D_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ the category obtained from $C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ by inverting Q_X^0 -quasi-isomorphisms in $C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$. Thus we have the functors

$$\begin{aligned} \lambda : D_1^b(\nu(\mathcal{O}_X), \text{Diff}_X) &\longrightarrow D^b(\nu(\mathcal{O}_X), \text{Diff}_X) \\ Q_{X,1}^0 : D_1^b(\nu(\mathcal{O}_X), \text{Diff}_X) &\longrightarrow D^b(\mathcal{P}_X) \end{aligned}$$

where $Q_{X,1}^0 := Q_X^0 \circ \lambda$.

4. De Rham Functor

4.1. Definition. Let $\mathcal{M} \in \nu(\mathcal{P}_X)$ and

$$\begin{aligned} \overline{\text{DR}}_X(\mathcal{M}) &:= \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M} = \\ (4.1.1) \quad &= [0 \longrightarrow \overset{0}{\mathcal{M}} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \overset{1}{\mathcal{M}} \longrightarrow \cdots \longrightarrow \Omega_X^d \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow 0]. \end{aligned}$$

The differentials are defined using the stratification map $s'_{\mathcal{M}}$ (see Proposition 2.2) and the projection $\mathcal{P}_X^1 \longrightarrow \Omega_X^1$. The complex $\overline{\text{DR}}_X(\mathcal{M})$ belongs to $C^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ and in particular it is also an object of $C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$. We define the functors

$$\begin{aligned} \overline{\text{DR}}_{1,X} : C^b(\mathcal{P}_X) &\longrightarrow C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X) \\ \mathcal{M}^\bullet &\longmapsto (\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)_{\text{tot}} =: \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet. \end{aligned}$$

and $\overline{\text{DR}}_X = \lambda_1 \circ \overline{\text{DR}}_{1,X}$.

We want to prove that this De Rham functor sends the multiplicative system of qis in $C^b(\mathcal{P}_X)$ into the multiplicative system of Q_X^0 -qis in $C^b(\nu(\mathcal{O}_X), \text{Diff}_X)$. In order to prove this result we need the following version of the crystalline Poincaré lemma $\mathcal{O}_X \xrightarrow{\text{qis}} Q_X^0 \overline{\text{DR}}_X(\mathcal{O}_X)$. This lemma may be found in [G, 6.5] and in [BeO, 6.12] where Berthelot Ogus proved a filtered version. We give here a simple proof of the result we need. We note that our proof also works well in characteristic p using the formalism of divided powers.

Let us remark:

4.2. Remark. [Be.O, 2.13]. The Pro-object $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ admits two different stratifications depending on the \mathcal{O}_X -module structure we chose on it. We consider on $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ its “left” \mathcal{O}_X -structure, (that given by p_0), (so its “right” \mathcal{O}_X -structure may be used in the tensor product with the De Rham complex). This is the construction of Grothendieck linearization. In this case $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$ is endowed with the stratification $\vartheta : \mathcal{P}_X^{m+n} \rightarrow \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{P}_X^m$ sending $(f \otimes g) \mapsto 1 \otimes g \otimes 1 \otimes f$.

On the other hand, if we consider the “right” structure on $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$, the stratification is given by the map δ .

4.3. Lemma. *The linearized De Rham complex is a resolution of \mathcal{O}_X*

$$\mathcal{O}_X \xrightarrow{d^0} Q_X^0 \overline{DR}_X(\mathcal{O}_X)$$

in $C^b(\mathcal{P}_X)$. In fact the complex

$$(4.3.1) \quad \mathcal{O}_X \xrightarrow{d_0} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \xrightarrow{\overline{\nabla}^0} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\overline{\nabla}^1} \cdots \xrightarrow{\overline{\nabla}^{d-1}} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \Omega_X^d$$

is exact and thus locally homotopic to zero since its terms are locally free.

Proof. First of all we remark that $Q_X^0 \overline{DR}_X(\mathcal{O}_X)$ is a complex in $C^b(\mathcal{P}_X)$ and the map $\mathcal{O}_X \xrightarrow{d^0} \{\mathcal{P}_X^m\}_{\mathbb{Z}}$ respects the stratifications (see the remark given below regarding the stratification on $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$). It is evident that $\overline{\nabla}^0 \circ d^0 = 0$.

The complex (4.3.1) as a complex of $\nu(\mathcal{P}_X)$ is represented by

$$(4.3.2.n)_{n \in \mathbb{Z}} \quad \{\mathcal{O}_X \xrightarrow{d_0} \mathcal{P}_X^n \xrightarrow{\overline{\nabla}_{(n)}^0} \mathcal{P}_X^{n-1} \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\overline{\nabla}_{(n)}^1} \cdots \xrightarrow{\overline{\nabla}_{(n)}^{d-1}} \mathcal{P}_X^{n-d} \otimes_{\mathcal{O}_X} \Omega_X^d\}_{n \in \mathbb{N}}.$$

We will prove by induction that (4.3.2.n) is exact for each $n \in \mathbb{N}$.

First of all we render the \mathcal{O}_X -linear differentials on the complex explicit by the use of the basis given by ξ_i (see remark 2.4). Then d^0 is the map

$$\begin{aligned} d^0 : \mathcal{O}_X &\longrightarrow \mathcal{P}_X^n \\ 1 &\longmapsto \mathbb{I} := 1 \otimes 1 \\ f &\longmapsto f \otimes 1 = f\mathbb{I} \end{aligned}$$

while $\overline{\nabla} := Q_X^0(\nabla)$ is the linearization of the De Rham complex ($\mathcal{O}_X \xrightarrow{\nabla^0} \Omega_X^1 \xrightarrow{\nabla^1} \cdots \xrightarrow{\nabla^{d-1}} \Omega_X^d$) obtained as

$$\begin{array}{ccc} \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \Omega_X^p & \xrightarrow{\overline{\nabla}_{(n)}^p} & \mathcal{P}_X^{n-1} \otimes_{\mathcal{O}_X} \Omega_X^{p+1} \\ \delta^{n,1} \otimes id_{\Omega_X^p} \downarrow & \nearrow id_{\mathcal{P}_X^{n-1}} \otimes \overline{\nabla}_{(1)}^p & \\ \mathcal{P}_X^{n-1} \otimes_{\mathcal{O}_X} \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \Omega_X^p & & \end{array}$$

where $\overline{\nabla}_{(1)}^p$ is

$$\begin{aligned} \overline{\nabla}_{(1)}^p : \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \Omega_X^p &\longrightarrow \Omega_X^{p+1} \\ (f \otimes g) \otimes \omega &\longmapsto f \nabla^p(g\omega) && \text{so that in local coordinates we have} \\ \mathbb{I} \otimes (\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) &\longmapsto 0 \\ \xi_i \otimes (\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) &\longmapsto \xi_i \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_p} && \forall i \in \{1, \dots, d\}. \end{aligned}$$

Now we proceed by induction on the “level” n in order to prove that (4.3.1) is exact. For $n = 0$ the complex (4.3.2.0) reduces to

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{id_{\mathcal{O}_X}} \mathcal{O}_X \longrightarrow 0 \longrightarrow 0$$

which is obviously exact. When $n = 1$ the complex (4.3.2.1) is

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d^0} \mathcal{P}_X^1 \longrightarrow \Omega_X^1 \longrightarrow 0$$

which is homotopic to zero via the \mathcal{O}_X -linear homotopism

$$\begin{array}{ccccc} \mathcal{P}_X^1 & \xrightarrow{q_1} & \mathcal{O}_X & \Omega_X^1 & \longrightarrow & \mathcal{P}_X^1 \\ f \otimes g & \longmapsto & fg & \xi_i & \longmapsto & \xi_i \end{array}$$

then $\mathcal{P}_X^1 \cong \mathcal{O}_X \oplus \Omega_X^1$. Let us suppose that the complex (4.3.2.n-1) is exact with $n \geq 1$. Then we

consider the diagram

$$\begin{array}{ccccccccc}
& 0 & & 0 & & 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{I}^n / \mathcal{I}^{n+1} & \longrightarrow & \mathcal{I}^{n-1} / \mathcal{I}^n \otimes_{\mathcal{O}_X} \Omega_X^1 & \longrightarrow & \dots & \longrightarrow & \mathcal{I}^{n-d} / \mathcal{I}^{n-d+1} \otimes_{\mathcal{O}_X} \Omega_X^d & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{P}_X^n & \longrightarrow & \mathcal{P}_X^{n-1} \otimes_{\mathcal{O}_X} \Omega_X^1 & \longrightarrow & \dots & \longrightarrow & \mathcal{P}_X^{n-d} \otimes_{\mathcal{O}_X} \Omega_X^d & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{P}_X^{n-1} & \longrightarrow & \mathcal{P}_X^{n-2} \otimes_{\mathcal{O}_X} \Omega_X^1 & \longrightarrow & \dots & \longrightarrow & \mathcal{P}_X^{n-d-1} \otimes_{\mathcal{O}_X} \Omega_X^d & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

whose columns are exact. By inductive hypothesis the third row is exact. Then the second row is exact if and only if the first is. So we will prove that the complex

$$(4.3.3.n) \quad 0 \longrightarrow \mathcal{I}^n / \mathcal{I}^{n+1} \xrightarrow{D^0} \mathcal{I}^{n-1} / \mathcal{I}^n \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{D^1} \dots \xrightarrow{D^{d-1}} \mathcal{I}^{n-d} / \mathcal{I}^{n-d+1} \otimes_{\mathcal{O}_X} \Omega_X^d \longrightarrow 0$$

is exact proving that its identity is homotopic to zero.

We have to construct \mathcal{O}_X -linear maps

$$s_p : \mathcal{I}^{n-p} / \mathcal{I}^{n-p+1} \otimes_{\mathcal{O}_X} \Omega_X^p \longrightarrow \mathcal{I}^{n-p+1} / \mathcal{I}^{n-p} \otimes_{\mathcal{O}_X} \Omega_X^{p-1}$$

for any $p = 1, \dots, d$ such that in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}^n / \mathcal{I}^{n+1} & \longrightarrow & \mathcal{I}^{n-1} / \mathcal{I}^n \otimes_{\mathcal{O}_X} \Omega_X^1 & \longrightarrow & \dots & \longrightarrow & \mathcal{I}^{n-d} / \mathcal{I}^{n-d+1} \otimes_{\mathcal{O}_X} \Omega_X^d & \longrightarrow & 0 \\
& & \downarrow id & \swarrow s_1 & \downarrow id & \swarrow s_2 & \downarrow id & \swarrow s_d & \downarrow id \\
0 & \longrightarrow & \mathcal{I}^n / \mathcal{I}^{n+1} & \longrightarrow & \mathcal{I}^{n-1} / \mathcal{I}^n \otimes_{\mathcal{O}_X} \Omega_X^1 & \longrightarrow & \dots & \longrightarrow & \mathcal{I}^{n-d} / \mathcal{I}^{n-d+1} \otimes_{\mathcal{O}_X} \Omega_X^d & \longrightarrow & 0
\end{array}$$

the identity of $\mathcal{I}^{n-p} / \mathcal{I}^{n-p+1} \otimes_{\mathcal{O}_X} \Omega_X^p$ would be $id = D^{p-1} \circ s_p + s_{p+1} \circ D^p$.

First we explicitly write the action of the differentials D^p on a basis:

$$\begin{aligned}
\mathcal{I}^{n-p} / \mathcal{I}^{n-p+1} \otimes_{\mathcal{O}_X} \Omega_X^p & \xrightarrow{D^p} \mathcal{I}^{n-p-1} / \mathcal{I}^{n-p} \otimes_{\mathcal{O}_X} \Omega_X^{p+1} \\
\xi_1^{\alpha_1} \dots \xi_d^{\alpha_d} \otimes \xi_{i_1} \wedge \dots \wedge \xi_{i_p} & \mapsto \sum_{j=1}^d \alpha_j \xi_1^{\alpha_1} \dots \xi_j^{\alpha_j-1} \dots \xi_d^{\alpha_d} \otimes \xi_j \wedge \xi_{i_1} \wedge \dots \wedge \xi_{i_p}
\end{aligned}$$

with $\alpha_1 + \dots + \alpha_d = n - p$.

We note that the map

$$\begin{array}{ccc}
\overbrace{\Omega_X^1 \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \Omega_X^1}^{p \text{ times}} & \longrightarrow & \overbrace{\Omega_X^1 \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \Omega_X^1}^{p \text{ times}} \\
\alpha_1 \otimes \dots \otimes \alpha_p & \mapsto & \sum_{\sigma \in \Sigma_p} (-1)^{sgn(\sigma)} \alpha_{\sigma(1)} \otimes \dots \otimes \alpha_{\sigma(p)}
\end{array}$$

induces a map $\sigma^p : \Omega_X^p \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \Omega_X^{p-1}$. We define s_p (up to the factor $n(p-1)!$) as the composition

$$\begin{array}{ccc}
\mathcal{I}^{n-p} / \mathcal{I}^{n-p+1} \otimes_{\mathcal{O}_X} \Omega_X^p & \xrightarrow{n(p-1)!s_p} & \mathcal{I}^{n-p+1} / \mathcal{I}^{n-p+2} \otimes_{\mathcal{O}_X} \Omega_X^{p-1} \\
\downarrow id \otimes \sigma^p & \nearrow m \otimes id & \\
\mathcal{I}^{n-p} / \mathcal{I}^{n-p+1} \otimes_{\mathcal{O}_X} \mathcal{I} / \mathcal{I}^2 \otimes_{\mathcal{O}_X} \Omega_X^{p-1} & &
\end{array}$$

where m is the map $m : \mathcal{I}^{n-p} / \mathcal{I}^{n-p+1} \otimes_{\mathcal{O}_X} \mathcal{I} / \mathcal{I}^2 \longrightarrow \mathcal{I}^{n-p+1} / \mathcal{I}^{n-p+2}$ induced by the multiplication. It is well defined because $\mathcal{I}^{n-p+1} \mathcal{I} = \mathcal{I}^{n-p+2} = \mathcal{I}^{n-p} \mathcal{I}^2$. We now explicitly calculate s_p on a local basis

$$\begin{aligned}
\mathcal{I}^{n-p} / \mathcal{I}^{n-p+1} \otimes_{\mathcal{O}_X} \Omega_X^p & \xrightarrow{s_p} \mathcal{I}^{n-p+1} / \mathcal{I}^{n-p+2} \otimes_{\mathcal{O}_X} \Omega_X^{p-1} \\
\xi_1^{\alpha_1} \dots \xi_d^{\alpha_d} \otimes \xi_{i_1} \wedge \dots \wedge \xi_{i_p} & \mapsto \frac{1}{n} \sum_{m=1}^p (-1)^{m+1} \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d} \xi_{i_m} \otimes \xi_{i_1} \wedge \dots \wedge \widehat{\xi_{i_m}} \wedge \dots \wedge \xi_{i_p}.
\end{aligned}$$

We note that this definition also makes sense in the divided powers setting. Indeed, in characteristic p we replace $\xi_i^{\alpha_i}$ by $\xi_i^{[\alpha_i]}$ and the local description becomes:

$$\begin{aligned} \mathcal{J}^{n-p}/\mathcal{J}^{n-p+1} \otimes_{\mathcal{O}_X} \Omega_X^p &\xrightarrow{s_p} \mathcal{J}^{n-p+1}/\mathcal{J}^{n-p+2} \otimes_{\mathcal{O}_X} \Omega_X^{p-1} \\ \xi_1^{[\alpha_1]} \cdots \xi_d^{[\alpha_d]} \otimes \xi_{i_1} \wedge \cdots \wedge \xi_{i_p} &\longmapsto \sum_{m=1}^p (-1)^{m+1} \xi_1^{[\alpha_1]} \cdots \xi_{i_m}^{[\alpha_{i_m}+1]} \xi_d^{[\alpha_d]} \otimes \xi_{i_1} \wedge \cdots \wedge \widehat{\xi_{i_m}} \wedge \cdots \wedge \xi_{i_p}. \end{aligned}$$

Now let us compute the composition $D^{p-1} \circ s_p + s_{p+1} \circ D^p$ on an element of the basis. We have

$$\begin{aligned} (D^{p-1} \circ s_p + s_{p+1} \circ D^p)(\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \otimes \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) &= \\ = D^{p-1} \left(\frac{1}{n} \sum_{m=1}^p (-1)^{m+1} \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \xi_{i_m} \otimes \xi_{i_1} \wedge \cdots \wedge \widehat{\xi_{i_m}} \wedge \cdots \wedge \xi_{i_p} \right) &+ \\ + s_{p+1} \left(\sum_{j=1}^d \alpha_j \xi_1^{\alpha_1} \cdots \xi_j^{\alpha_j-1} \cdots \xi_d^{\alpha_d} \otimes \xi_j \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_p} \right) &= \\ = \frac{1}{n} \left(\sum_{m=1}^p \sum_{j=1, j \neq i_m}^d (-1)^{m+1} \alpha_j \xi_1^{\alpha_1} \cdots \xi_j^{\alpha_j-1} \cdots \xi_d^{\alpha_d} \xi_{i_m} \otimes \xi_j \wedge \xi_{i_1} \wedge \cdots \wedge \widehat{\xi_{i_m}} \wedge \cdots \wedge \xi_{i_p} \right) &+ \\ + \frac{1}{n} \left(\sum_{m=1}^p (\alpha_{i_m} + 1) \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \otimes \xi_{i_1} \wedge \cdots \wedge \xi_{i_p} \right) &+ \\ + \frac{1}{n} \left(\sum_{m=1}^p \sum_{j=1, j \neq i_m}^d (-1)^m \alpha_j \xi_1^{\alpha_1} \cdots \xi_j^{\alpha_j-1} \cdots \xi_d^{\alpha_d} \xi_{i_m} \otimes \xi_j \wedge \xi_{i_1} \wedge \cdots \wedge \widehat{\xi_{i_m}} \wedge \cdots \wedge \xi_{i_p} \right) &+ \\ + \frac{1}{n} \left(\sum_{j=1 \neq i_1, \dots, i_p}^d \alpha_j \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \otimes \xi_{i_1} \wedge \cdots \wedge \xi_{i_p} \right) &= \\ = \frac{1}{n} \left(p + \sum_{j=1}^d \alpha_j \right) \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \otimes \xi_{i_1} \wedge \cdots \wedge \xi_{i_p} &= \\ = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \otimes \xi_{i_1} \wedge \cdots \wedge \xi_{i_p} \end{aligned}$$

as desired. \square

The complex $\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \Omega_X^\bullet \cong \overline{\mathrm{DR}}_X(\{\mathcal{P}_X^m\}_{\mathbb{Z}})$ where we take $\{\mathcal{P}_X^m\}_{\mathbb{Z}} = p_1^*(\mathcal{O}_X)$. Thus we obtain a ‘‘Poincaré Lemma’’: $\mathcal{O}_X \cong \overline{\mathrm{DR}}_X(\{\mathcal{P}_X^m\}_{\mathbb{Z}})$. This result is the ‘‘dual’’ of the quasi-isomorphism $\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}_X \cong \omega_X[d]$ [B; VI.3.5].

The same result holds true on taking $\{\mathcal{P}_X^m\}_{\mathbb{Z}} = p_0^*(\mathcal{O}_X)$. In this case we obtain the following corollary.

4.4. Corollary. *The complex*

$$(4.4.1) \quad \mathcal{O}_X \xrightarrow{d_1} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \longrightarrow \cdots \longrightarrow \Omega_X^d \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}}$$

is exact. Hence,

$$\mathcal{O}_X \longrightarrow \overline{\mathrm{DR}}(p_0^*(\mathcal{O}_X))$$

is a quasi-isomorphism in $C^b(\nu(\mathcal{P}_X))$.

4.5. Corollary. *Let $\mathcal{M} \in \nu(\mathcal{P}_X)$. The complexes*

$$(4.5.1) \quad \mathcal{M} \xrightarrow{d_1} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega_X^d \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{M}$$

and

$$(4.5.2) \quad \mathcal{M} \xrightarrow{d_0} \mathcal{M} \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \Omega_X^1 \longrightarrow \cdots \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \Omega_X^d$$

are exact.

Proof. Let us consider the sequence (4.5.2). Its analogue for $\mathcal{M} = \mathcal{O}_X$ is the sequence 4.3.1, which is exact and so locally homotopic to zero. Any additive functor respects homotopies, so 4.3.1 tensorized over \mathcal{O}_X with \mathcal{M} gives 4.5.2 which is exact. \square

5. Equivalences of Categories

5.1. Remark. Let \mathcal{F}^\bullet be an object in $C_1(\nu(\mathcal{O}_X), \text{Diff}_X)$. By definition the differential $d_{\mathcal{F}}^i : \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$ is a morphism of Pro-object represented by differential operators of order at most one. It defines in a unique way a morphism $\bar{d}_{\mathcal{F}}^i : \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$. This morphism extends in a unique way to a morphism of stratified Pro-modules $\mathbf{Q}_X^0(d_{\mathcal{F}}^i) : \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^i \longrightarrow \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^{i+1}$. Using the \mathcal{O}_X -base of $\mathcal{P}_{X,1}$ given in local coordinates by $\mathbb{I}, dx_1, \dots, dx_d$ we have for any section s of \mathcal{F}^i

$$(5.1.1) \quad \begin{aligned} d_{\mathcal{F}}^i(s) &= \bar{d}_{\mathcal{F}}^i(\mathbb{I} \otimes s) \\ d_{x_j}^i(s) &:= \bar{d}_{\mathcal{F}}^i(\xi_j \otimes s) \end{aligned}$$

(where the second is taken as definition of $d_{x_j}^i$). The maps $d_{x_j}^i : \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$ are maps of \mathcal{O}_X -modules depending on the choice of the coordinates.

5.2. Definition. Let $\mathcal{F}^\bullet \in C^b(\nu(\mathcal{O}_X), \text{Diff}_X)$. Then \mathcal{F}^\bullet is a Ω_X^\bullet -module in Pro-objects. Let by definition $\sigma_{\mathcal{F}}^{i,j} : \Omega_X^j \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} \longrightarrow \mathcal{F}^i$ be the Ω_X^\bullet -structural maps. They are:

$$\sigma_{\mathcal{F}}^{i,0} = \text{id}_{\mathcal{F}^i} : \mathcal{F}^i \longrightarrow \mathcal{F}^i$$

$$\begin{array}{ccc} \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{i-1} & \xrightarrow{\bar{d}_{\mathcal{F}}^{i-1}} & \mathcal{F}^i \\ \uparrow & \nearrow \sigma_{\mathcal{F}}^{i,1} & \\ \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{i-1} & & \end{array}$$

and in general

$$\begin{array}{ccc} \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} & \xrightarrow{\text{id}_{\mathcal{P}_X^1} \otimes \bar{d}_{\mathcal{F}}^{i-2}} \cdots \xrightarrow{\bar{d}_{\mathcal{F}}^{i-1}} & \mathcal{F}^i \\ \uparrow & \nearrow \sigma_{\mathcal{F}}^{i,j} & \\ \Omega_X^j \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} & & \end{array}$$

where the last vertical map

$$\Omega_X^j \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} \longrightarrow \overbrace{\mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{P}_X^1}^{j \text{ times}} \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j}$$

is induced by the shuffle on a basis.

Now we prove a technical lemma which will be used in the proof of our main theorem.

5.3. Lemma. Given $\mathcal{F}^\bullet \in C_1(\nu(\mathcal{O}_X), \text{Diff}_X)$; the morphisms $d_{\mathcal{F}}^i, d_{x_j}^i$ of (5.1.1) for $i \in \mathbb{Z}$ and $j \in \{0, \dots, d\}$ satisfy the following conditions:

- i) $d_{\mathcal{F}}^{i+1} \circ d_{\mathcal{F}}^i = 0$
- ii) $d_{x_j}^{i+1} \circ d_{\mathcal{F}}^i + d_{\mathcal{F}}^{i+1} \circ d_{x_j}^i = 0$
- iii) $d_{x_j}^{i+1} \circ d_{x_k}^i + d_{x_k}^{i+1} \circ d_{x_j}^i = 0$

iv) $d_{x_j}^{i+1} \circ d_{x_j}^i = 0$.

Proof. The first condition is given by the hypothesis $\mathcal{F}^\bullet \in C_1(\nu(\mathcal{O}_X), \text{Diff}_X)$, so the composition $d_{\mathcal{F}}^{i+1} \circ d_{\mathcal{F}}^i = 0$. In order to prove ii) to iv) we remark that:

$$\begin{aligned} d_{\mathcal{F}}^{i+1} \circ d_{x_j}^i(s) + d_{x_j}^{i+1} \circ d_{\mathcal{F}}^i(s) &= \overline{d_{\mathcal{F}}^{i+1} d_{\mathcal{F}}^i}(\xi_j \otimes s) = 0 \\ d_{x_j}^{i+1} \circ d_{x_k}^i(s) + d_{x_k}^{i+1} \circ d_{x_j}^i(s) &= \overline{d_{\mathcal{F}}^{i+1} d_{\mathcal{F}}^i}(\xi_j \xi_k \otimes s) = 0 \\ d_{x_j}^{i+1} \circ d_{x_j}^i(s) + d_{x_j}^{i+1} \circ d_{\mathcal{F}}^i(s) &= \overline{d_{\mathcal{F}}^{i+1} d_{\mathcal{F}}^i}(\xi_j^2 \otimes s) = 0 \end{aligned}$$

where $\xi_j = 1 \otimes x_j - x_j \otimes 1 \in \mathcal{P}_X^1$. We recall that $\overline{d_{\mathcal{F}}^{i+1} d_{\mathcal{F}}^i}$ is obtained by the composition

$$\begin{array}{ccccc} \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^i & \xrightarrow{\text{id}_{\mathcal{P}_X^1} \otimes \overline{d_{\mathcal{F}}^i}} & \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{i+1} & \xrightarrow{\overline{d_{\mathcal{F}}^{i+1}}} & \mathcal{F}^{i+2} \\ \delta^{2,1} \otimes \text{id}_{\mathcal{F}^i} \uparrow & & \nearrow \overline{d_{\mathcal{F}}^{i+1} d_{\mathcal{F}}^i} & & \\ \mathcal{P}_X^2 \otimes_{\mathcal{O}_X} \mathcal{F}^i & & & & \end{array}$$

and $\delta^{2,1}$ was defined in 2.4. □

5.4. Definition. Let dx_1, \dots, dx_n be a local basis for Ω_X^1 (where n is the dimension of X). We define the maps

$$\begin{aligned} \eta_{\mathcal{F}}^{i,j} : \quad \Omega_X^j \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} &\longrightarrow \mathcal{F}^i \\ f dx_{i_1} \wedge \dots \wedge dx_{i_j} \otimes s &\longmapsto f dx_{i_1}^{i-1} \circ \dots \circ dx_{i_j}^{i-j}(s) \end{aligned}$$

where f is a section of \mathcal{O}_X and s is a section of \mathcal{F}^{i-j} . This definition does not depend on local coordinates and moreover $\sigma_{\mathcal{F}}^{i,j} = j! \eta_{\mathcal{F}}^{i,j}$.

5.5. Definition. Let $q = \{q_m\}_{\mathbb{Z}} : \{\mathcal{P}_X^m\}_{\mathbb{Z}} \longrightarrow \mathcal{O}_X$ be the usual projection which is linear for both the \mathcal{O}_X -module structures of $\{\mathcal{P}_X^m\}_{\mathbb{Z}}$. Given $\mathcal{F}^\bullet \in C_1(\nu(\mathcal{O}_X), \text{Diff}_X)$ we define the morphisms

$$\Phi_{\mathcal{F}}^i : \bigoplus_{j=0}^d \Omega_X^j \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} \longrightarrow \mathcal{F}^i$$

for each $i \in \mathbb{Z}$ in the following way: we consider the composition

$$\begin{array}{ccc} \Omega_X^j \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} & \xrightarrow{\text{id}_{\Omega_X^j} \otimes q \otimes \text{id}_{\mathcal{F}^{i-j}}} & \Omega_X^j \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} \\ & \searrow \Phi_{\mathcal{F}}^{i,j} & \downarrow \eta_{\mathcal{F}}^{i,j} \\ & & \mathcal{F}^i \end{array}$$

and by definition $\Phi_{\mathcal{F}}^i := \sum_{j=0}^d \Phi_{\mathcal{F}}^{i,j}$.

5.6. Theorem. We have two morphisms of functors

$$\Psi : \text{id}_{C^b(\mathcal{P}_X)} \longrightarrow Q_X^0 \circ \overline{\text{DR}}_X = Q_{1,X}^0 \circ \overline{\text{DR}}_{1,X}$$

(functors between $C^b(\mathcal{P}_X)$ and itself)

$$\Phi : \overline{\text{DR}}_{1,X} \circ Q_X^0 \longrightarrow \text{id}_{C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)}$$

(functors between $C_1^b(\nu(\mathcal{O}_X), \text{Diff}_X)$ and itself). They induce quasi-isomorphisms of complexes. So the functors $\overline{\text{DR}}_{1,X}$ and $\overline{\text{DR}}_X$ localize with respect to Q_X^0 -quasi-isomorphisms inducing the functor

$$\begin{aligned} \overline{\text{DR}}_{1,X} : D^b(\mathcal{P}_X) &\longrightarrow D_1^b(\nu(\mathcal{O}_X), \text{Diff}_X). \\ \overline{\text{DR}}_X : D^b(\mathcal{P}_X) &\longrightarrow D^b(\nu(\mathcal{O}_X), \text{Diff}_X). \end{aligned}$$

Moreover $\overline{\mathrm{DR}}_X$ (resp. $\overline{\mathrm{DR}}_{1,X}$) is an equivalence of categories whose quasi-inverse is the functor Q_X^0 (resp. $\mathrm{Q}_{1,X}^0$).

Proof. The morphism $d^0 : \mathcal{O}_X \rightarrow \mathcal{P}_X^m$ induces a morphism of bicomplexes (see (4.5.2))

$$\Psi_{\mathcal{M}^\bullet} : \mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \Omega_X^\bullet.$$

Then we obtain a morphism of complexes

$$\begin{aligned} \mathcal{M}^\bullet &\longrightarrow \mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \Omega_X^\bullet \cong \\ &\cong \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \Omega_X^\bullet \cong \\ &\cong \mathrm{Q}_X^0(\overline{\mathrm{DR}}_X(\mathcal{M}^\bullet)) \cong \\ &\cong \mathrm{Q}_{1,X}^0(\overline{\mathrm{DR}}_{1,X}(\mathcal{M}^\bullet)). \end{aligned}$$

The isomorphism between the first and the second complex is induced by the stratification on \mathcal{M}^\bullet which is the isomorphism $\{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet \cong \mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}}$. By Corollary 4.5 we obtain that it is a quasi-isomorphism. So the functors $\overline{\mathrm{DR}}_{1,X}$ and $\overline{\mathrm{DR}}_X$ send qis in $D^b(\mathcal{P}_X^\bullet)$ into Q_X^0 -qis which permits us to localize them.

We have to prove that the diagram

$$\begin{array}{ccc} \Omega_X^j \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^{i-j} & \xrightarrow{d_{\overline{\mathrm{DR}}_X^0}^i} & \Omega_X^j \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^{i+1-j} \\ \Phi_{\mathcal{F}}^i \downarrow & & \downarrow \Phi_{\mathcal{F}}^{i+1} \\ \mathcal{F}^i & \xrightarrow{d_{\mathcal{F}}^i} & \mathcal{F}^{i+1} \end{array}$$

is commutative.

The complex $\overline{\mathrm{DR}}_X \mathrm{Q}_X^0(\mathcal{F}^\bullet) = (\mathcal{G}^{\bullet\bullet})_{\mathrm{tot}}$ where

$$\mathcal{G}^{p,q} = \Omega_X^p \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^q$$

and

$$d_{\mathcal{G}}^{p,q} : \Omega_X^p \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^q \longrightarrow \Omega_X^{p+1} \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^q$$

is

$$(5.6.1) \quad d_{\mathcal{G}}^{p,q} = d_{\overline{\mathrm{DR}}_X(p_0^*(\mathcal{O}_X))}^p \otimes id_{\mathcal{F}^q}$$

while

$$d_{\mathcal{G}}^{p,q} : \Omega_X^p \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^q \longrightarrow \Omega_X^p \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^{q+1}$$

is

$$(5.6.2) \quad d_{\mathcal{G}}^{p,q} = id_{\Omega_X^p} \otimes \mathrm{Q}_X^0(d_{\mathcal{F}}^q);$$

where $\overline{\mathrm{DR}}_X(p_0^*(\mathcal{O}_X))$ was considered in 4.4.1.

Given $I^{\bullet,\bullet}$ a bounded bicomplex with commuting differentials $d_I^{p,q} : I^{p,q} \rightarrow I^{p+1,q}$ and $d_I^{p,q} : I^{p,q} \rightarrow I^{p,q+1}$, the total complex associated to it is denoted by I_{tot}^\bullet with $I_{\mathrm{tot}}^r := \bigoplus_{p+q=r} I^{p,q}$ and $d_{I_{\mathrm{tot}}}(x) = d_I^{p,q}(x) + (-1)^p d_I^{p,q+1}(x)$ for any $x \in I^{p,q}$.

The set $\mathrm{Hom}(A_{\mathrm{tot}}^\bullet, B^\bullet)$ of morphisms of complexes between a total complex of a bicomplex and a complex is the set families of maps $\{\varphi^p : A_{\mathrm{tot}}^p \rightarrow B^p\}_p$ such that $d_B^p \circ \varphi^p = \varphi^{p+1} \circ d_{A_{\mathrm{tot}}}^p$. Then $\mathrm{Hom}(A_{\mathrm{tot}}^\bullet, B^\bullet)$ is isomorphic to the set of families of maps $\{\varphi^{p,q} : A^{q,p-q} \rightarrow B^p\}_{p,q}$ satisfying the following conditions

$$(5.6.3) \quad d_B^p \circ \varphi^{p,q} = \varphi^{p+1,q+1} d_A^{q,p-q} + (-1)^q \varphi^{p+1,q} \circ d_A^{q,p-q}$$

for any p, q . So we have only to prove that

$$(5.6.4) \quad d_{\mathcal{F}}^p \circ \Phi_{\mathcal{F}}^{p,q} = \Phi_{\mathcal{F}}^{p+1,q+1} d_{\mathcal{G}}^{q,p-q} + (-1)^q \Phi_{\mathcal{F}}^{p+1,q} \circ d_{\mathcal{G}}^{q,p-q}$$

is true.

It is enough to check these relations locally, choosing local coordinates x_1, \dots, x_n . The sections $f dx_{i_1} \wedge \dots \wedge dx_{i_q} \otimes \mathbb{I} \otimes s$ generate $\Omega_X^q \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^{p-q}$ where s is a section of \mathcal{F}^{p-q} . Then

$$\begin{aligned} d_{\mathcal{F}}^p \circ \Phi_{\mathcal{F}}^{p,q}(f dx_{i_1} \wedge \dots \wedge dx_{i_q} \otimes \mathbb{I} \otimes s) &= d_{\mathcal{F}}^p(f d_{x_{i_1}}^{p-1} \dots d_{x_{i_q}}^{p-q}(s)) = \\ &= f d_{\mathcal{F}}^p d_{x_{i_1}}^{p-1} \dots d_{x_{i_q}}^{p-q}(s) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} d_{x_i}^p d_{x_{i_1}}^{p-1} \dots d_{x_{i_q}}^{p-q}(s) \end{aligned}$$

while

$$\begin{aligned} \Phi_{\mathcal{F}}^{p+1,q+1} d_{\mathcal{G}}^{q,p-q}(f dx_{i_1} \wedge \dots \wedge dx_{i_q} \otimes \mathbb{I} \otimes s) &= \Phi_{\mathcal{F}}^{p+1,q+1} \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_q} \otimes \mathbb{I} \otimes s \right) = \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i} d_{x_i}^p d_{x_{i_1}}^{p-1} \dots d_{x_{i_q}}^{p-q}(s) \end{aligned}$$

For the last term we have

$$\begin{aligned} \Phi_{\mathcal{F}}^{p+1,q} d_{\mathcal{G}}^{q,p-q}(f dx_{i_1} \wedge \dots \wedge dx_{i_q} \otimes \mathbb{I} \otimes s) &= \Phi_{\mathcal{F}}^{p+1,q}(f dx_{i_1} \wedge \dots \wedge dx_{i_q} \otimes \mathbb{I} \otimes d_{\mathcal{F}}^{p-q}(s)) = \\ &= f d_{x_{i_1}}^p \dots d_{x_{i_q}}^{p-q+1} d_{\mathcal{F}}^{p-q}(s) \end{aligned}$$

Thus, using Lemma 5.3, we prove our assertion. Moreover the composition

$$\begin{array}{ccc} \mathcal{F}^\bullet & \xrightarrow{d^1} & \Omega_X^\bullet \otimes_{\mathcal{O}_X} \{\mathcal{P}_X^m\}_{\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet \\ & \searrow id_{\mathcal{F}^\bullet} & \downarrow \Phi_{\mathcal{F}}^\bullet \\ & & \mathcal{F}^\bullet \end{array}$$

is the identity so $\Phi_{\mathcal{F}^\bullet}$ is a \mathbb{Q}_X^0 -quasi-isomorphism. In particular the functor $\overline{\mathrm{DR}}_X$ localizes to $\overline{\mathrm{DR}}_X : D^b(\mathcal{P}_X) \longrightarrow D^b(\nu(\mathcal{O}_X), \mathrm{Diff}_X)$ and it is an equivalence of categories with quasi-inverse the localized \mathbb{Q}_X^0 functor. \square

5.7. Corollary. *The functor*

$$\lambda : D_1^b(\nu(\mathcal{O}_X), \mathrm{Diff}_X) \longrightarrow D^b(\nu(\mathcal{O}_X), \mathrm{Diff}_X)$$

is an equivalence of categories whose quasi-inverse is the functor $\overline{\mathrm{DR}}_X \circ \mathbb{Q}_X^0$.

6. Crystals in Pro-Modules

We refer to Grothendieck exposé [G] for the definition of crystalline site $\mathrm{Cris}(X/\mathbb{C})$ in characteristic zero (and also to Berthelot's thesis [Be]). We denote by $\mathcal{O}_{X/\mathbb{C}}$ the sheaf on $\mathrm{Cris}(X/\mathbb{C})$ such that for any object (nilpotent closed immersion $U \hookrightarrow T$ with $U \subset X$ open subset) its value is $\mathcal{O}_{X/\mathbb{C}}(U \hookrightarrow T) := \mathcal{O}_T$. It is a ringed sheaf on $\mathrm{Cris}(X/\mathbb{C})$.

6.1. Definition. *A crystal in Pro-modules $\{\mathcal{F}_i\}_{i \in I}$ is a sheaf on $\mathrm{Pro}\text{-}\mathcal{O}_{X/\mathbb{C}}$ -modules in the crystalline site $\mathrm{Cris}(X/\mathbb{C})$ such that for any morphism $p : \{U' \hookrightarrow T'\} \rightarrow \{U \hookrightarrow T\}$ given by the diagram*

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ \downarrow & & \downarrow p \\ U & \longrightarrow & T \\ \downarrow & & \\ X & & \end{array}$$

we have $\{p^(\mathcal{F}_i(U \hookrightarrow T))\}_{i \in I} \cong \{\mathcal{F}_i(U' \hookrightarrow T')\}_{i \in I}$.*

6.2. Theorem. *The category $\nu(\mathcal{P}_X)$ is equivalent to the category of crystals in Pro-coherent $\mathcal{O}_{X/\mathbb{C}}$ -modules.*

Proof. The proof is equivalent to the classical proof of the equivalence between stratified $\mathcal{O}_{X/\mathbb{C}}$ -modules and crystals (see [Be]). If $\{\mathcal{F}_i\}_{i \in I}$ is a crystal in Pro-coherent $\mathcal{O}_{X/\mathbb{C}}$ -modules then $\{\mathcal{F}_i(X \xrightarrow{id_X} X)\}_{i \in I} \in \nu(\mathcal{O}_X)$. Taking the diagram defined by the diagonal thickenings $X_n \rightarrow X \times X$ with $n \in \mathbb{N}$

$$\begin{array}{ccc} X & \longrightarrow & X_n \\ id \downarrow & & \downarrow p_0 \quad p_1 \\ X & \xrightarrow{id} & X \end{array}$$

we obtain the stratification $\{p_0^*(\mathcal{F}_i(X \hookrightarrow X_n))\}_{i \in I} \cong \{p_1^*(\mathcal{F}_i(X \hookrightarrow X_n))\}_{i \in I}$ in the Pro-object $\{\mathcal{F}_i(X \xrightarrow{id_X} X)\}_{i \in I}$.

Conversely if $\{\mathcal{F}_i\}_I \in \nu(\{\mathcal{P}_X^m\}_{\mathbb{Z}})$ we define a sheaf on $Cris(X/\mathbb{C})$ in the following way. For any object $U \hookrightarrow T \in Cris(X/\mathbb{C})$ there exists locally a section $h : T \rightarrow X$ (X is smooth). We define $CR(\{\mathcal{F}_i\}_I)(U \hookrightarrow T) := \{h^*(\mathcal{F}_i)\}_I$. These local definitions patch together to define a sheaf in Pro-coherent $\mathcal{O}_{X/\mathbb{C}}$ -Modules which is a crystal. \square

6.3. Remark. Deligne proved in a conference at IHES (1970 unpublished) that the category of “regular” crystals in Pro-modules on X is equivalent to the category of “algebraic” constructible sheaves in the analytic space X^{an} . Hence, Theorem 5.6 might be interpreted as an algebraic Pro-version of a Riemann-Hilbert equivalence.

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